Two-loop Loewner potentials and CFT partition functions

Sid Maibach

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UNIVERSITÄT BONN IAM SFB 1060

Loewner potentials

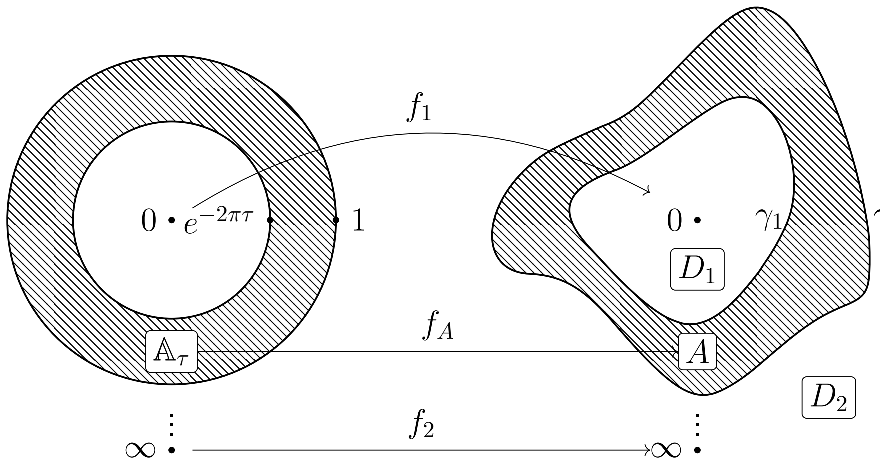
- The Loewner potential $\mathcal{H}_\Sigma(\vec{\gamma})$ is a **conformally invariant action functional** for measures on simple curves or loops in a Riemann surface Σ. For loops ε-close to given $\vec{\gamma}$ and $\vec{\eta}$:

$$\lim_{\varepsilon\to 0}\frac{\mu_\Sigma^{\mathbf{c}}(\mathcal{O}_\varepsilon(\vec{\gamma}))}{\mu_\Sigma^{\mathbf{c}}(\mathcal{O}_\varepsilon(\vec{\xi}))} = e^{\frac{\mathbf{c}}{2}(\mathcal{H}_\Sigma(\vec{\gamma})-\mathcal{H}_\Sigma(\vec{\xi}))}.$$

- The most general definition is as a ratio of **zeta-regularized determinants** of Laplacians [PW23]:

$$\mathcal{H}_\Sigma(\vec{\gamma}) = \log\frac{\det_\zeta \Delta_{g|_\Sigma}}{\prod_{A\in\pi_0(\Sigma\setminus\vec{\gamma})}\det_\zeta \Delta_{g|_A}}$$

- For a single loop in the Riemann sphere, the measure is the **Schramm-Loewner evolution** (SLE) loop measure [Zha21].



- Our results express the two-loop case in terms of normalized **Brownian loop measure** or **pre-Schwarzian derivatives**:

$$\begin{aligned}\mathcal{H}_{\hat{\mathbb{C}}}(\gamma_1,\gamma_2) &= \mathcal{H}_{\hat{\mathbb{C}}}(\gamma_1)+\mathcal{H}_{\hat{\mathbb{C}}}(\gamma_2)+\mathbf{\Lambda}^*(\gamma_1,\gamma_2)+(\text{const.})\\ &= \mathcal{H}_{\hat{\mathbb{C}}}(e^{-2\pi\tau}S^1,S^1) - \tfrac{1}{3}\log\left|\frac{f_2'(\infty)}{f_1'(0)}\right| + \\ &\tfrac{1}{12\pi}\Big(\int_{e^{-2\pi\tau}\mathbb{D}}|\boldsymbol{\mathcal{A}}[f_1]|^2\,|\mathrm{d}z|^2 + \int_{\mathbb{A}_\tau}|\boldsymbol{\mathcal{A}}[f_A]|^2\,|\mathrm{d}z|^2 + \int_{\mathbb{D}^*}|\boldsymbol{\mathcal{A}}[f_2]|^2\,|\mathrm{d}z|^2\Big)\end{aligned}$$

Problem: No lower bound

1. Fixing the modulus τ of the annulus between the two loops, minimizers of $\mathcal{H}_{\hat{\mathbb{C}}}(\gamma_1,\gamma_2)$ are given by pairs of **circles**.
2. Minimizing among all circles, we find that

$$\mathcal{H}_{\hat{\mathbb{C}}}(e^{-2\pi\tau}S^1,S^1)\to-\infty,$$

as $\tau\to\infty$ (the circles move further apart).

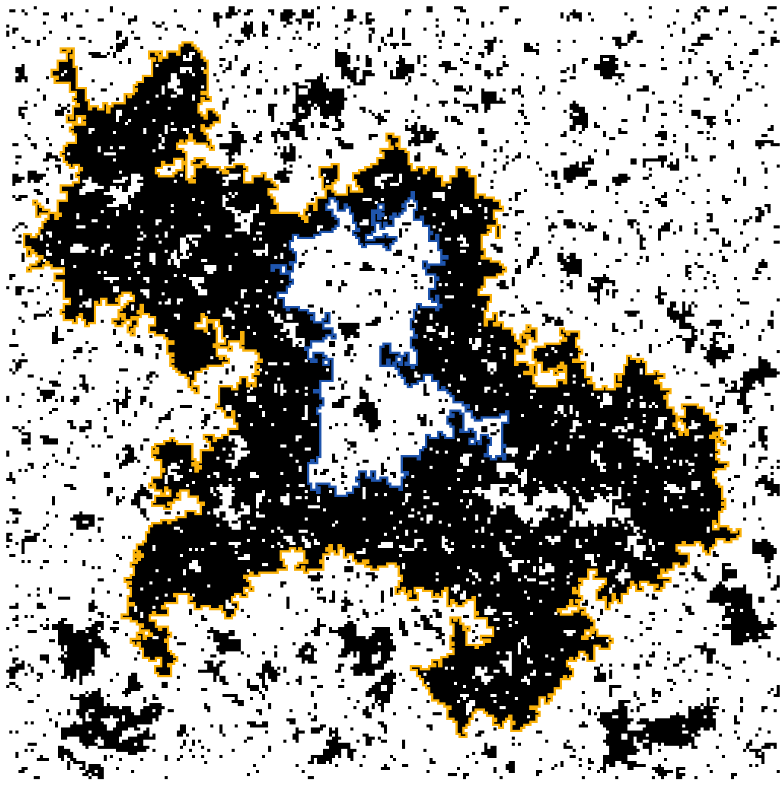
Solution: Annulus partition functions

- $\mathcal{H}_{\hat{\mathbb{C}}}(\gamma_1,\gamma_2)$ and $\mathcal{H}^Z_{\hat{\mathbb{C}}}(\gamma_1,\gamma_2)$ differ by a function of τ.
- A minimizing configuration exists if and only if

$$e^{-\frac{\pi}{3}\mathbf{c}\tau}Z_{\mathrm{d}z\mathrm{d}\bar{z}}(\mathbb{A}_\tau)$$

has a global minimum $\tau\in(0,\infty)$.
- **Question:** For which CFT does this hold?

Interfaces in conformal field theory



Critical Ising model with two interfaces ± 1 highlighted.

- The loops γ_1 and γ_2 are supposed to appear as interfaces in configurations of a fixed (2D, Euclidean) conformal field theory (CFT).
- For example, consider the **scaling limit** of a statistical mechanics model with states σ, local energy functional $S(\sigma)$, and critical temperature β.
- Heuristically, the **probability** of γ_1 and γ_2 being interfaces is

$$\sum_{\substack{\sigma \text{ such that}\\ \gamma_1,\,\gamma_2 \text{ are interfaces}}}\frac{e^{-\beta S(\sigma)}}{Z(\hat{\mathbb{C}})} = \sum_{\substack{\sigma|_{D_1},\,\sigma|_A,\,\sigma|_{D_2}\\ \text{satisfying}\\ \text{boundary conditions}}}\frac{e^{-\beta\left(S(\sigma|_{D_1})+S(\sigma|_A)+S(\sigma|_{D_2})\right)}}{Z(\hat{\mathbb{C}})}$$

$$= \frac{Z(D_1)Z(A)Z(D_2)}{Z(\hat{\mathbb{C}})}.$$

The **partition functions** come with boundary conditions as in Cardy's work on bCFT [Car08].

- This suggests to generalize the Loewner potential as

$$\mathcal{H}^Z_{\hat{\mathbb{C}}}(\gamma_1,\gamma_2) = \frac{2}{\mathbf{c}}\log\frac{Z_g(D_1)Z_g(A)Z_g(D_2)}{Z_g(\hat{\mathbb{C}})}.$$

We also derive this geometrically from the **real determinant line bundle** as defined in [KS07].

References

[Car08] J. Cardy. Boundary Conformal Field Theory. 2008.

[KS07] M. Kontsevich and Y. Suhov. On Malliavin measures, SLE, and CFT. 2007.

[PW23] E. Peltola and Y. Wang. Large deviations of multichordal SLE_{0+}, real rational functions, and zeta-regularized determinants of Laplacians. 2023.

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2025 University of Bonn, Institute for Applied Mathematics
maibach@uni-bonn.de, www.sidmaibach.eu